On a numerical truncation approximation algorithm for transfer matrix method

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A numerical truncation technique is described for reducing the numerical instability problems associated with the utilization of the transfer matrix method, especially in cases where the frequency of ultrasound, the number of layers, or the thickness of the layers become very large. This rather simplistic modification to the numerical coding extends the transfer matrix method to a wide range of applications, without any complex and computationally intensive reformulation. © *2000 Acoustical Society of America.* [S0001-4966(00)04102-3]

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INTRODUCTION

Modeling of acoustic wave propagation in layered elastic media is important because of the extensive range of problems which can be addressed. A popular and widely reported method for modeling wave propagation in layered structures is the transfer matrix method, which has been credited to Thomson[1] and has been well documented.[2–14]

It is a well-known fact that one of the characteristics of the technique is the occurrence of numerical instabilities, even for layered isotropic structures. The numerical instabilities have been observed especially when the overall thickness of the structure or the frequency of the harmonic ultrasonic wave becomes very high or when the intermediate elastic supports are very stiff.[9,10] The numerical difficulties also increase at higher oblique incidence angles (near grazing angles). They are reported to be dependent on the "*fdr*," parameter which is a product of the frequency (f), thickness (d), and number of ply groups (r).[12] Here, ply groups is defined as the superlayer repetitions (a superlayer being a combination of ply groups that can repeat or have mirror images). Thus, for thick laminates (high *fdr* values) having a large number of ply groups, the transfer matrix method does not provide a reliable model. This results in a limitation of the model application to very thin layered structures composed of simple stacking sequences. Moreover, inverse problems like stacking sequence identification and elastic constant inversion become increasingly more difficult to solve.

The cause of these computational problems is traced to the loss of precision that occurs when performing normal arithmetic calculations using computers of limited precision. In the transfer matrix method, the global transfer matrix is related to the local transfer matrices by

$$A=\prod_{k=1}^{k=n} A_k,$$

where A_k is the local transfer matrix of the kth layer of an n-layered laminate. In this method, the relatively insignificant inaccuracies in computation are amplified by the large number of matrix premultiplications when computing the global transfer matrix A from the local transfer matrices A_k, as in the case of a large number of ply groups. Furthermore, the matrices A_k contain terms like $e^{i\alpha\xi d_k}$, where α is the ratio of the wave number component in the thickness direction to that along the surface in the incident plane, i.e., a partial wave solution; ξ is the wave number in the wave number component along the surface in the incident plane; and d_k is the thickness of the kth layer. Here, small precision errors are amplified by the exponential term, especially for high thicknesses.

There are several publications which have addressed the stability issue and suggested improvements to reduce the instability. For instance, Dunkin[8] had developed a delta operator technique which has since been improved by Kundu and Mal,[10] Levesque and Piche,[11] and Castaings and Hosten.[12] This method uses up to 20th-order delta matrices and is computationally intensive. The method involves computing subdeterminants of local transfer matrices and is aimed at preventing the accumulation of precision errors during computation of the global transfer matrix. Most work[6] has been concentrated on isotropic materials utilizing the delta-2 operator. However, Castaings and Hosten[12] have adapted the technique to anisotropic media (composite laminates) where the delta-3 operator has been introduced. Delta operators do indeed provide a much-improved transfer matrix formulation, but nevertheless, they result in significantly increasing computational time as a result of a large number of extra subdeterminant computations. The computational time on a Sun SPARC Server was determined to increase by 30–40 times for a typical thick composite laminate case study. Speed of computation is even more critical during the use of inverse techniques for determining the material properties, where the procedure has to be frequently repeated during a typical inverse search process.[13] Furthermore, as reported by Castaings and Hosten,[12] closed-form analytical expressions for the local transfer matrix elements and subdeterminants need to be found using a mathematical software that performs symbolic operations. This leads to the development of a considerably new formulation of the technique which can sometimes result in very complex coding for numerical

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TABLE I. Material properties of the graphite–epoxy composite used in the analysis (Ref. 7).

Viscoelastic graphite/epoxy ($\rho = 1500\ \mathrm{kg/m^3}$)									
Elastic constant C_{ij} (GPa)	C_{11}	C_{22}	C_{33}	C_{12}	C_{13}	C_{23}	C_{44}	C_{55}	C_{66}
	132.00	12.30	12.10	6.90	5.90	5.50	3.32	6.21	6.15
Viscoelastic constant η_{ij} (GPa-μs)	η_{11}	η_{22}	η_{33}	η_{12}	η_{13}	η_{23}	η_{44}	η_{55}	η_{66}
	0.400	0.037	0.043	0.001	0.016	0.021	0.009	0.015	0.020

analysis for both the forward and the inverse solutions. For thick structures with periodic repetition of layers, Floquet wave techniques may be applicable.[13]

I. NUMERICAL TRUNCATION ALGORITHM

An approximation algorithm by numerically truncating the higher end values was attempted, with considerable success without compromising computational speed. Loss of accuracy and validity of the results is always a concern during any truncation process. The truncation algorithm discussed here provided results which compare very well with the results obtained using the delta operator technique[12] and was found to be stable for a wide range of case studies.[14,15] The technique is simple, almost trivial, and no significant change in the traditional analytical expressions for the transfer matrix method is required. In this remedy, the error amplification caused by the exponential terms was limited by setting a maximum threshold value on the real part of the exponent $(i\alpha\xi d_k)$. To account for the ply-group dependence, the maximum threshold was set to $25/r$. Hence, the following condition was imposed:

$$\text{IF REAL } (i\alpha\xi d_k) > 25/r \text{ THEN REAL } (i\alpha\xi d_k) = 25/r.$$

Here, ξ is equal to $(2\pi f)/c$, where f is the frequency and c is the phase velocity of the partial mode. Hence, the *fdr* parameter, which influences the onset of instabilities, is inherent in the numerical truncation algorithm. The selection of 25 as the threshold was based on several numerical convergence studies conducted on graphite–epoxy composite laminates. A numerical approach to solving the instability problems with the transfer matrix method has also been suggested elsewhere.[16]

In this letter, only fluid-loaded laminates constructed from fiber-reinforced composite materials (graphite–epoxy) were considered. The reflection and transmission factor characteristics for a longitudinal wave obliquely incident on the laminate will be employed in order to demonstrate the capability of a numerical truncation technique. Numerical experiments were carried out on the viscoelastic graphite/epoxy material used by Deschamps and Hosten.[7] The viscoelastic material properties are listed in Table I. The three commonly encountered stacking sequences, *viz.*, unidirectional, cross-ply, and quasi-isotropic, were investigated. Reflection factor versus incidence angle θ plots in the $\phi = 0°$ incidence plane were studied because numerical problems arise at oblique incidence. All multioriented laminates were assumed to be composed of equal-thickness ply groups, i.e., $d_k = d(\text{total laminate thickness})/r$. All routines were coded in FORTRAN 77 and executed on SUN workstations using double precision and double complex data types.

The behavior of reflection factor, with and without numerical truncation, for a unidirectional laminate was examined in a case study where numerical instabilities have been reported earlier.[7] The frequency and laminate thickness parameters were those used in Deschamps and Hosten,[7] i.e., frequency $(f) = 2.242$ MHz and thickness $(d) = 3.434$ mm. The plots of the reflection factor versus incidence angle without and with numerical truncation are shown in Fig. 1(a) and

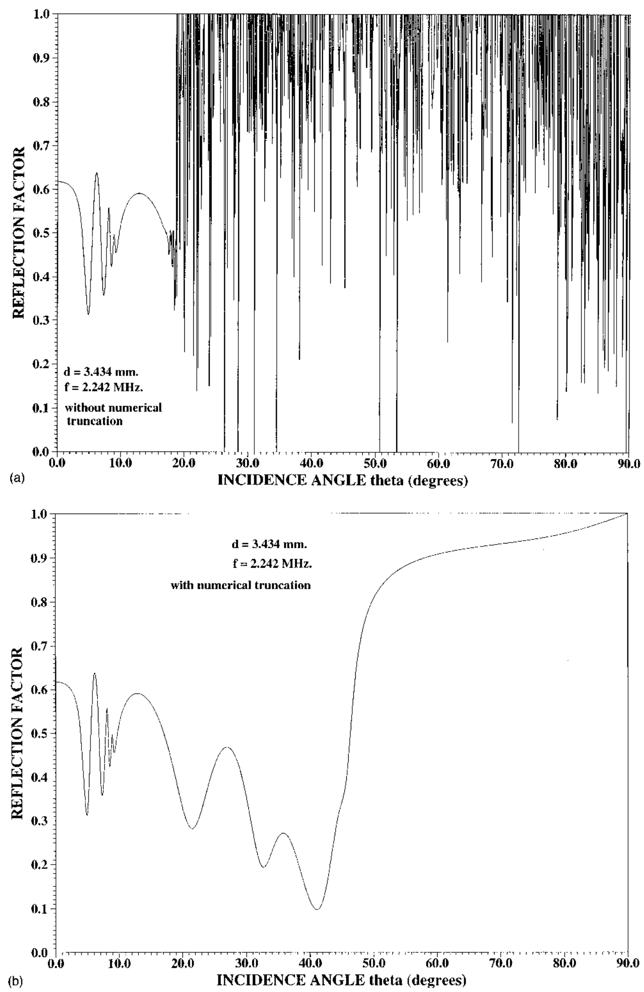

FIG. 1. (a) Computed reflection factor response without any numerical truncation for a viscoelastic unidirectional graphite–epoxy composite laminate with thickness $(d) = 3.434$ mm, at frequency $(f) = 2.241$ MHz at a plane of incidence along the fibers. (b) Computed reflection factor response with numerical truncation correction for a viscoelastic unidirectional graphite–epoxy composite laminate with thickness $(d) = 3.434$ mm, at frequency $(f) = 2.241$ MHz at a plane of incidence along the fibers.

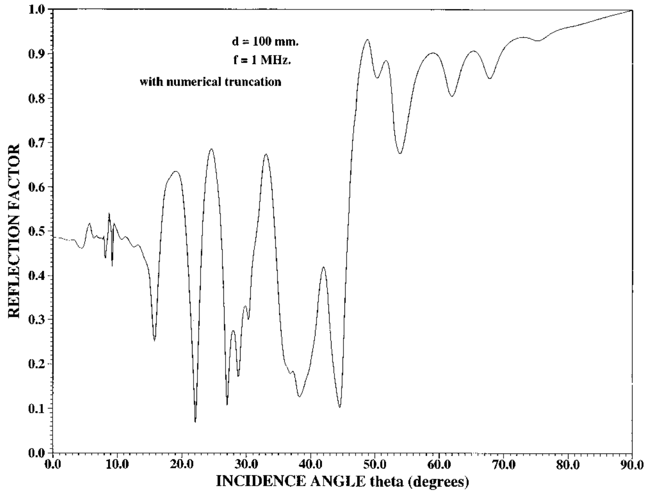

FIG. 2. Computed reflection factor response with numerical truncation for a very thick viscoelastic cross-ply $[0/90]_{4S}$ graphite–epoxy composite laminate with thickness $(d) = 100$ mm, at frequency $(f) = 1.00$ MHz at a plane of incidence along the fibers.

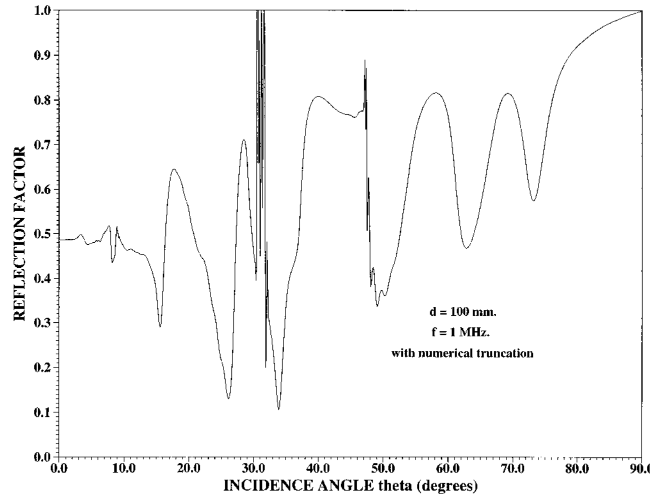

FIG. 4. Computed reflection factor response with numerical truncation for a very thick viscoelastic $[0/90]_{8S}$ graphite–epoxy composite laminate with thickness $(d) = 100$ mm, at frequency $(f) = 1.00$ MHz at a plane of incidence along the fibers. This shows some numerical instabilities near 30-deg angle of incidence.

(b), respectively. As can be observed, without numerical truncation [Fig. 1(a)], the transfer matrix formulation is computationally stable till around an incidence angle $\theta = 18°$, after which instabilities creep in. On the other hand, incorporating numerical truncation [Fig. 1(b)] stabilizes the computation, with the reflection factor being computed right up to $\theta = 90°$. Furthermore, computation with truncation agrees extremely well with the result published by Deschamps and Hosten[7] for the same case—especially the experimental one at high angles of incidence and for several other case studies.[15] In fact, their numerical computation, in spite of using an improved transfer matrix algorithm, does not match well with experiments at incidence angles greater than 50°.

Next, the stability of the formulation with truncation was examined for a very thick composite laminate. The case study involved 100-mm-thick laminated plates at an incident frequency $f = 1$ MHz $(fd = 100$ MHz mm). Figures 2–4 show the reflection factor plots for $[0°/90°]_{4S}$, $[0°/90°/+45°/-45°]_{4S}$, $[0°/90°]_{8S}$ laminates. It can be noted from Figs. 2–4 that the reflection factor computation is stable, while the results obtained without employing the numerical truncation were highly unstable. However, from Fig. 4, for $[0°/90°]_{8S}$ where the number of ply-group repetitions $r = 16$ is doubled, as compared to $[0°/90°]_{4S}$ where $r = 8$, some instabilities seem to appear around $\theta = 30°$. Nevertheless, when the laminate thickness was reduced to $d = 50$ mm $(fd = 50$ MHz mm), no instabilities were observed. This demonstrates that the numerical truncation may not be stable for all values of *fdr*. Yet, based on the laminate layups and high *fd* values used in the case studies here, representing extreme cases for many practical composite structures, the numerical truncation significantly reduces the computational instabilities.

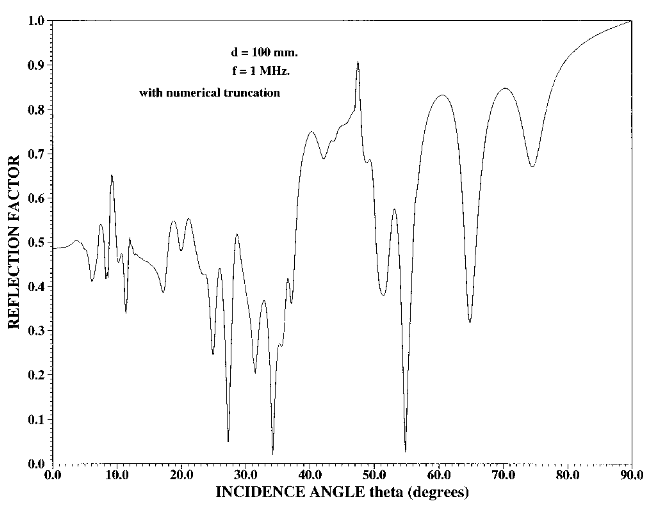

FIG. 3. Computed reflection factor response with numerical truncation for a very thick viscoelastic $[0/90/+45/-45]_{4S}$ graphite–epoxy composite laminate with thickness $(d) = 100$ mm, at frequency $(f) = 1.00$ MHz at a plane of incidence along the fibers.

II. CONCLUSION

The numerical truncation method, discussed in this letter, involves limiting the exponential terms involved to a certain threshold in order to prevent instabilities due to precision inaccuracies and computational limitation in handling large numbers. Alternative methods such as the delta operator technique have been shown to solve this problem, but these technique require extensive reformulation and significantly add to the computational time. In contrast, the numerical truncation algorithm is simpler and does not compromise computational speed. Numerical truncation was examined for unidirectional, cross-ply, and quasi-isotropic viscoelastic laminates. It was observed that despite some limitations on the thickness, *vis-a-vis* ply-group repetitions, the method can be applied successfully to a wider range of laminate stacking sequences, ultrasonic frequencies, and thicknesses where the traditional transfer matrix method demonstrates instabilities.

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